Transverse Wave Motion part 2

21st SEPTEMBER 2020

Standing waves on a string of a fixed length

- The displacement on the string at any point is given by $y = ae^{i(\omega t - kx)} + be^{i(\omega t + kx)}$
- With the boundary condition that y = 0 at x = 0 and x = *l* at all times, thus

$$y = (-2i)ae^{i(\omega t)} \sin kx$$

- The complete expression for the displacement of the ***nth harmonic*** is given by

$$y_n = 2a(-i)(\cos \omega_n t + i \sin \omega_n t) \sin \frac{\omega_n x}{c}$$

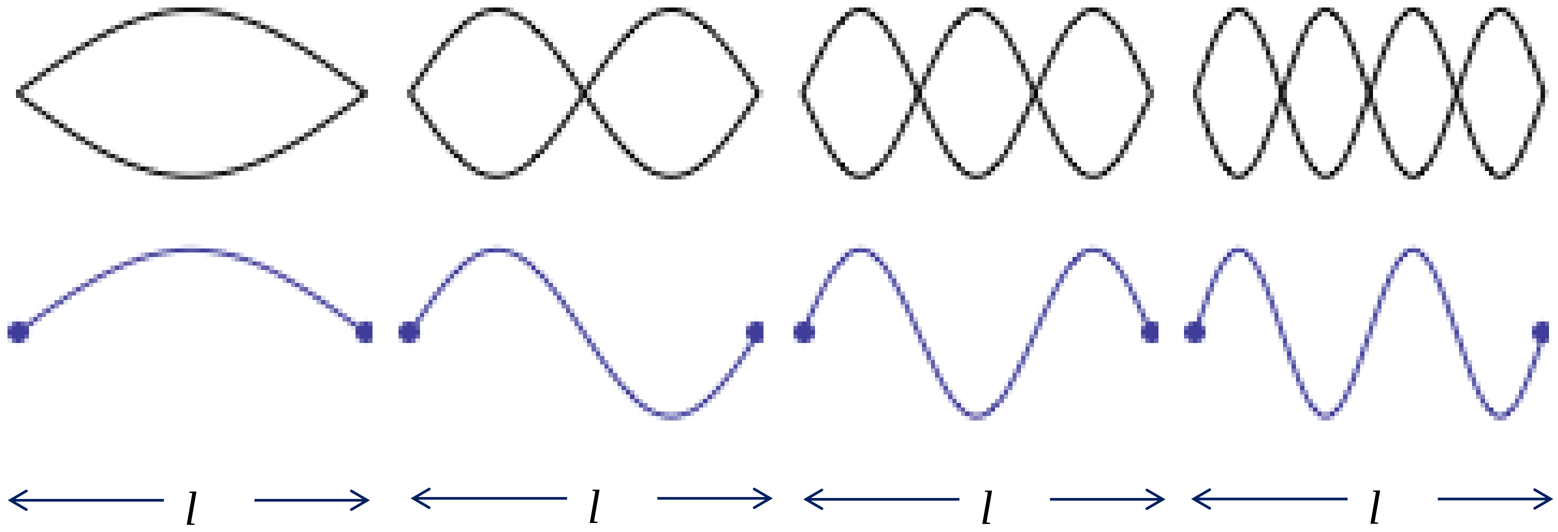
- This may be expressed as $y_n = (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{\omega_n x}{c}$

- Where the amplitude of nth mode is given by $(A_n + B_n)^{1/2} = 2a$

NOTE ω_n is the natural frequency of the n^{th} normal mode and given as $n\pi c/L$

Normal mode condition

- Due to the second boundary condition: y = 0 at x = *l* at all times, $kl = n\pi$.
- This leads to $l = {}^{n\lambda}/_{2}$ which is the condition for each normal mode of standing waves.
- The animation below shows the first four allowed mode shapes for a fixed-fixed string. The number of "humps" (antinodes) corresponds to the value of *n*.



https://www.acs.psu.edu/drussell/Demos/Pluck-Fourier/Pluck-Fourier.html

Energy of each normal modes of vibrating string

- The energy in each harmonic is composed of kinetic and potential energy.

$$E_n(\text{kinetic + potential}) = E_n(\text{kinetic}) + E_n(\text{potential})$$

$$= \frac{1}{2}\int_0^l \rho \dot{y}_n^2 dx + \frac{1}{2} T \int_0^l \left(\frac{\partial y_n}{\partial x}\right)^2 dx$$

$$= \frac{1}{4}\rho l \omega_n^2 \left(A_n^2 + B_n^2\right)$$

$$= \frac{1}{4} m \omega_n^2 \left(A_n^2 + B_n^2\right)$$

- Where $y_n = \left(A_n \cos \omega_n t + B_n \sin \omega_n t\right) \sin \frac{\omega_n x}{c}$

Standing Wave Ratio (SWR)

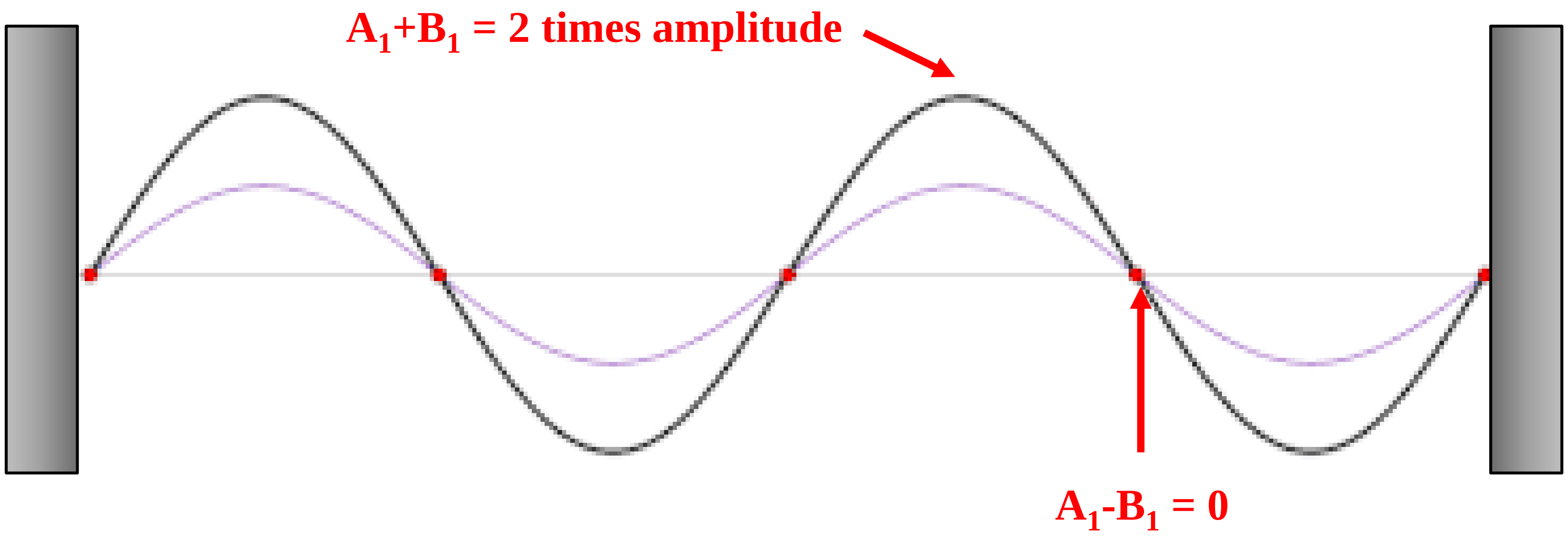
- If a progressive wave system is partially reflected from a boundary, the incident and reflected amplitudes partially cancel out.
- In this case the ratio of **maximum to minimum amplitudes** in the standing wave system is called

$$\text{Standing Wave Ratio (SWR)} = \frac{|A_1| + |B_1|}{|A_1| - |B_1|} = \frac{1+|r|}{1-|r|}$$

NOTE : A_1 = incident amplitude
B_1 = reflected amplitude

- Where $|r| = |B_1/A_1|$: the magnitude of the amplitude ratio;
- Since $0 < |r| < 1$ therefore $1 < \text{SWR} < \infty$
- SWR is always greater than or equal to unity and can be used to determine the sample's **reflection coefficient (*r*)**, its **absorption coefficient (α)** and its **impedance (Z)**.

Standing wave ratio and reflection coefficient



Incident wave is fully reflected out of phase at both ends, creating a (black) standing wave. $r = -1$, SWR $= \infty$

What is an application of SWR?

Example 1

- Finding the superposition of given travelling waves on a string which is fixed at both ends.
- $y_1(x,t) = A\cos(\omega t - kx)$ and $y_2(x,t) = rA\cos(\omega t + kx)$: where r is the coefficient of amplitude reflection.
- Note that the superposition of the travelling waves represents the displacement of a wave on the string.

Solution

- By applying an appropriate trigonometric identity, the superposition is simply found to be

$$y(x,t) = A\cos(\omega t - kx) + rA\cos(\omega t + kx)$$

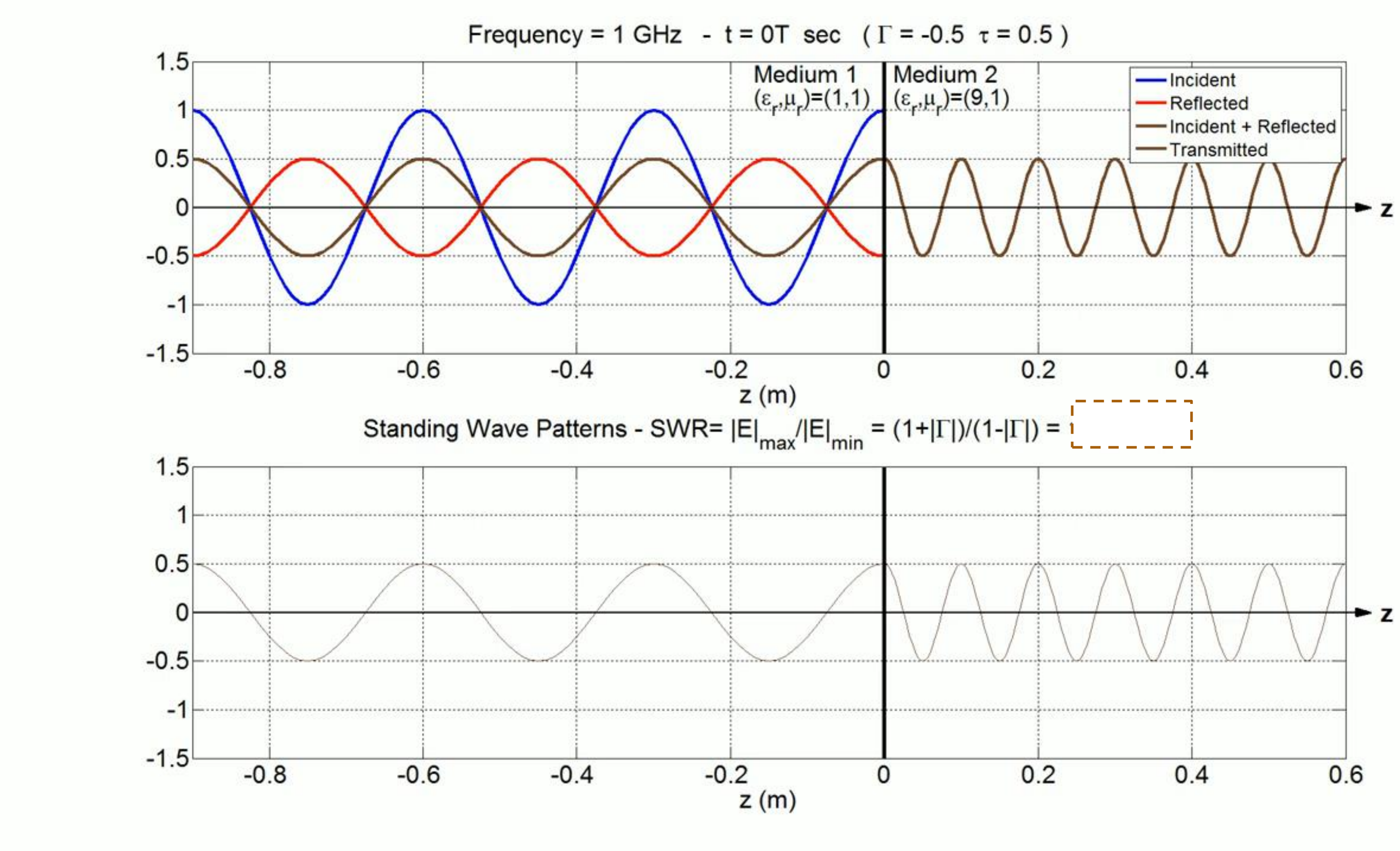
$$= A\cos\omega t\cos kx + A\sin\omega t\sin kx + rA\cos\omega t\cos kx - rA\sin\omega t\sin kx$$

$$= \underbrace{A(1+r)\cos\omega t\cos kx}_{\text{standing wave}} + \underbrace{A(1-r)\sin\omega t\sin kx}_{\text{standing wave}}$$

Standing wave   standing wave

- This clearly shows that the displacement is actually the superposition of standing waves.

Example 2



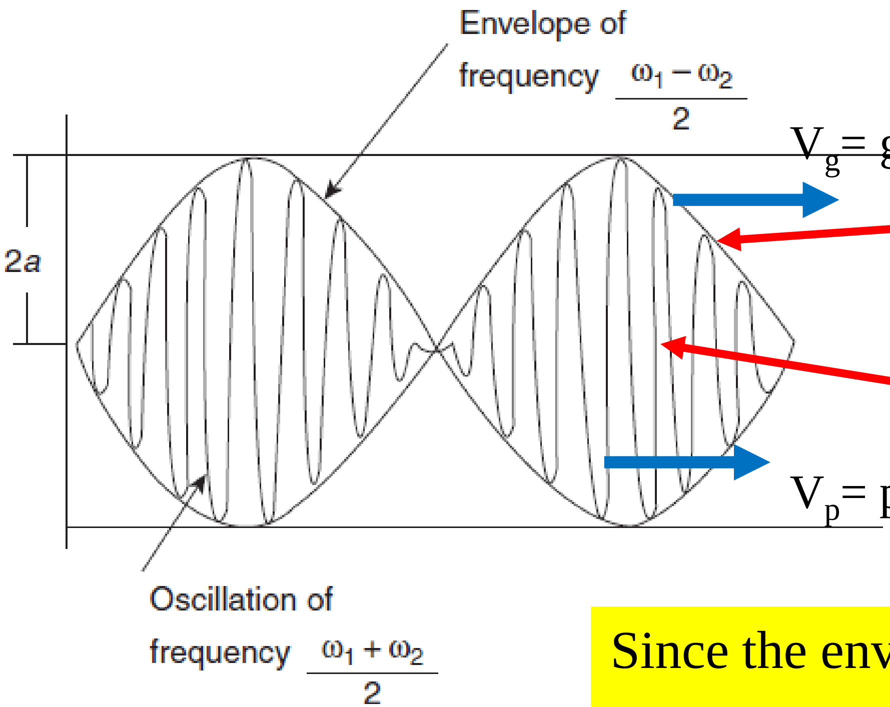
https://www.youtube.com/watch?v=s5MBno0PZjE

Wave group

Superposition of two travelling waves of almost equal frequencies;

$$y_1 = a\,\mathbf{cos}\left(\omega_1 t - k_1 x\right),\, y_2 = a\,\mathbf{cos}\left(\omega_2 t - k_2 x\right)$$

Superposition of amplitude and phase gives



V_g= group velocity

V_p= phase velocity

$$y = y_1 + y_2$$

$$= 2a\cos\left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right]$$

$$\times\cos\left[\frac{(\omega_1 + \omega_2)t}{2} - \frac{(k_1 + k_2)x}{2}\right]$$

Since the envelope is a representation of the wave group and described by the 1st cosine function, the velocity so called group velocity is defined as $\frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta\omega}{\Delta k}$

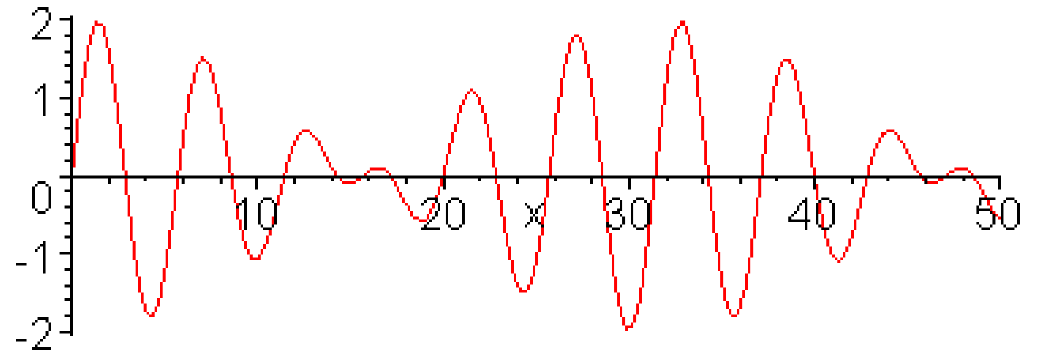
Group velocity

Consider 2 cases which are the **same phase velocities** and **different phase velocities**.

(1) The two waves have the same phase velocities; i.e., $\omega_1/k_1 = \omega_2/k_2 = c$. This leads to

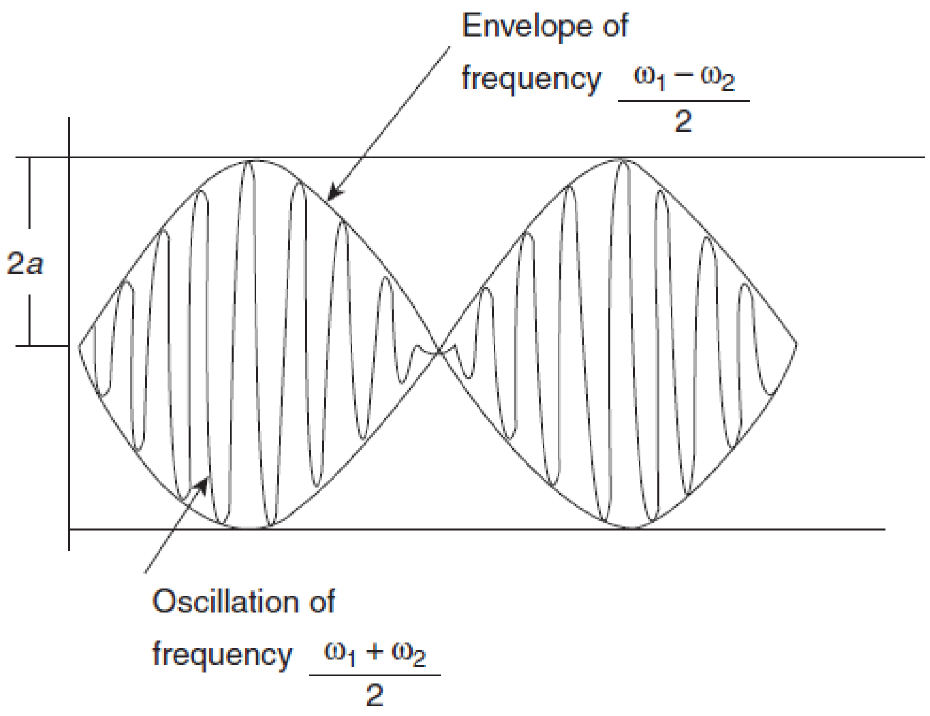
$$\text{group velocity}: \frac{\Delta\omega}{\Delta k} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = c\frac{(k_1 - k_2)}{k_1 - k_2} = c$$

This suggests that the component frequencies and their superposition, or group will travel with the **same velocity**, the profile of their combination remaining constant.

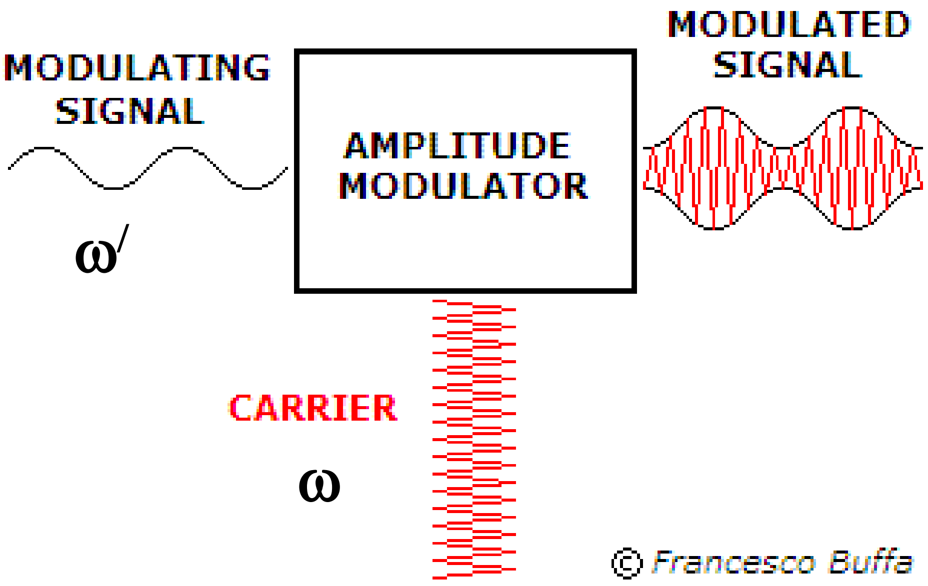


This situation, i.e. $v_g = v_p$, is for non-dispersive waves.

Revisit the superposition of two waves of almost equal frequencies

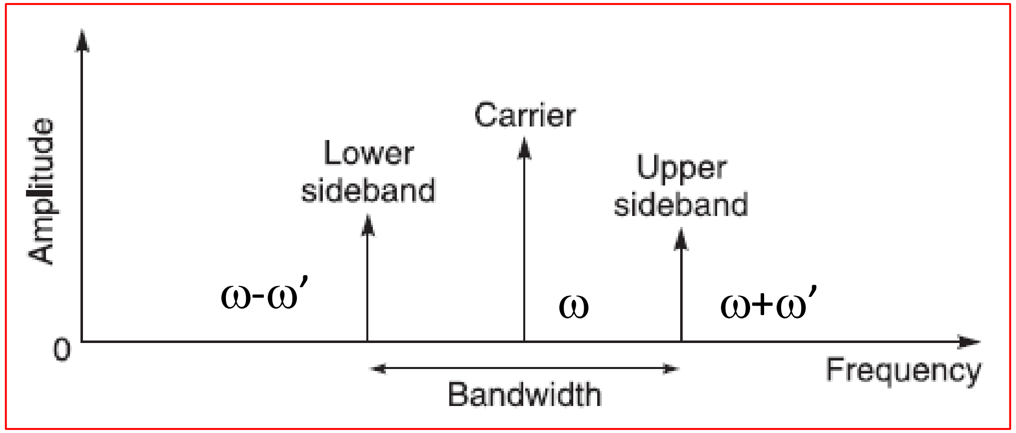


- The superposition produces an amplitude which varies between 2a and 0.
- This situation is called complete or 100% modulation.
- The high frequency wave is **amplitude modulated**.
- This actually is an example of sent signal found in a communication technique for transmitting information.



Amplitude Modulation (AM)

- A general expression of an amplitude modulated wave is given as $y = A\cos(\omega t - kx)$
- Where the modulated amplitude $A = a + b\cos\omega' t$
- $\omega > \omega'$
- This gives
$$y = a\cos(\omega t - kx) + \frac{b}{2}\left\{\left[\cos(\omega + \omega')t - kx\right] + \left[\cos(\omega - \omega')t - kx\right]\right\}$$
- Two new frequencies $\omega \pm \omega'$ (sidebands) are introduced.



https://dsp.stackexchange.com/questions/47604/when-listening-in-to-an-am-signals-of-various-frequencies-how-do-we-exactly-tun

https://giphy.com/gifs/analog-BzYOp24AXnHEI

Group velocity (contd.)

(2) Two frequencies components have **different phase velocities** so that $\omega_1/k_1 \neq \omega_2/k_2$.

The group velocity is given as

$$\frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta\omega}{\Delta k}$$

This situation, i.e. $v_g \neq v_p$, is for dispersive waves.

- The superposition of the two waves will **no longer remain constant and the group profile will change with time**.
- If a group contains a number of components of frequencies which are nearly equal, the original expression for the group velocity is written

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

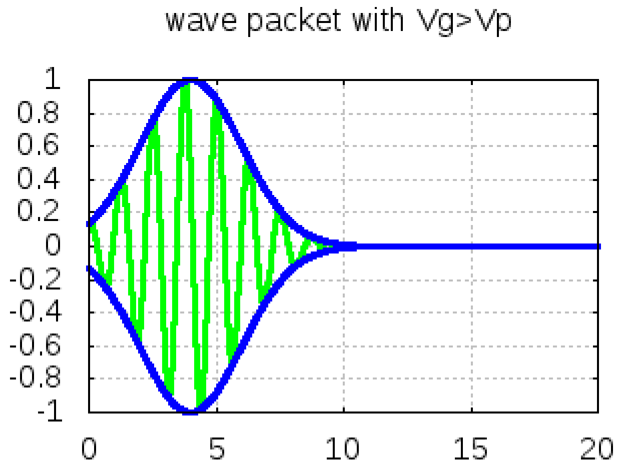
Dispersion relation

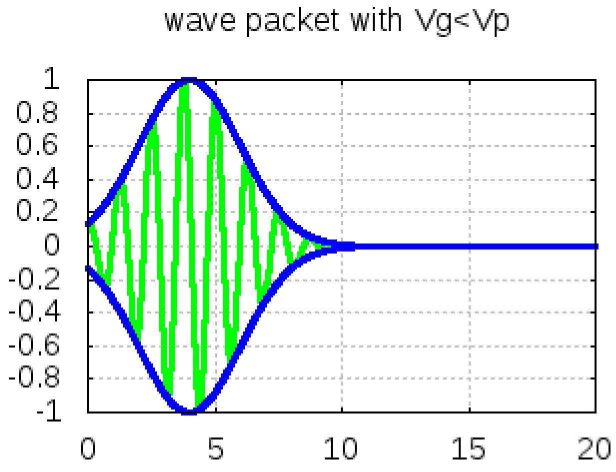
- A medium at which the phase velocity is frequency dependent (ω/k not constant) is known as a dispersive medium and a **dispersion relation** expresses the variation of ω as a function of k.
- This leads to three possible changes with time of the group velocity profile relative to the phase velocity.
- They include

(1) $v_g = v_p$: a non-dispersive relation

(2) $v_g < v_p$: a normal dispersion relation

(3) $v_g > v_p$: an anomalous dispersion relation



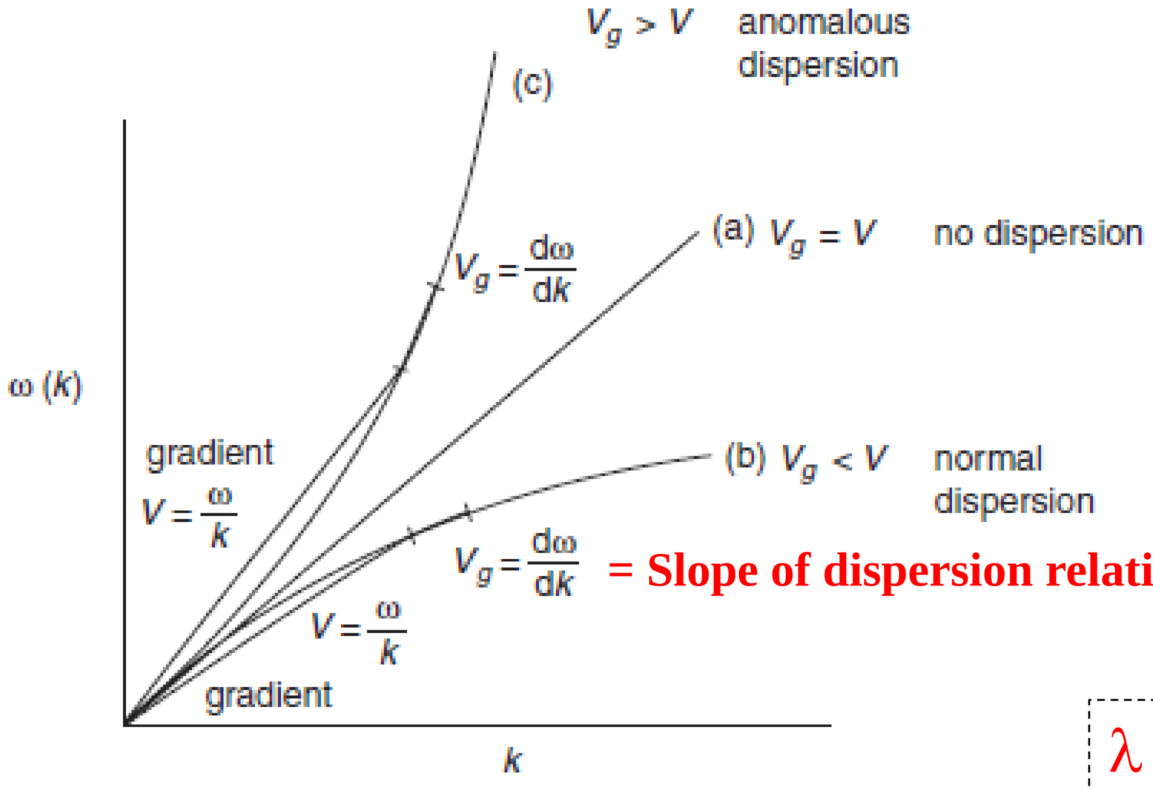


http://theory.ipp.ac.cn/~yj/free_software.html

Example 3

- Draw a dispersion relation of electromagnetic waves in vacuum.

Illustration of dispersion relation



= Slope of dispersion relation

- Since $\omega = kv$; v = *phase velocity*

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(kv) = v + k\frac{dv}{dk}$$

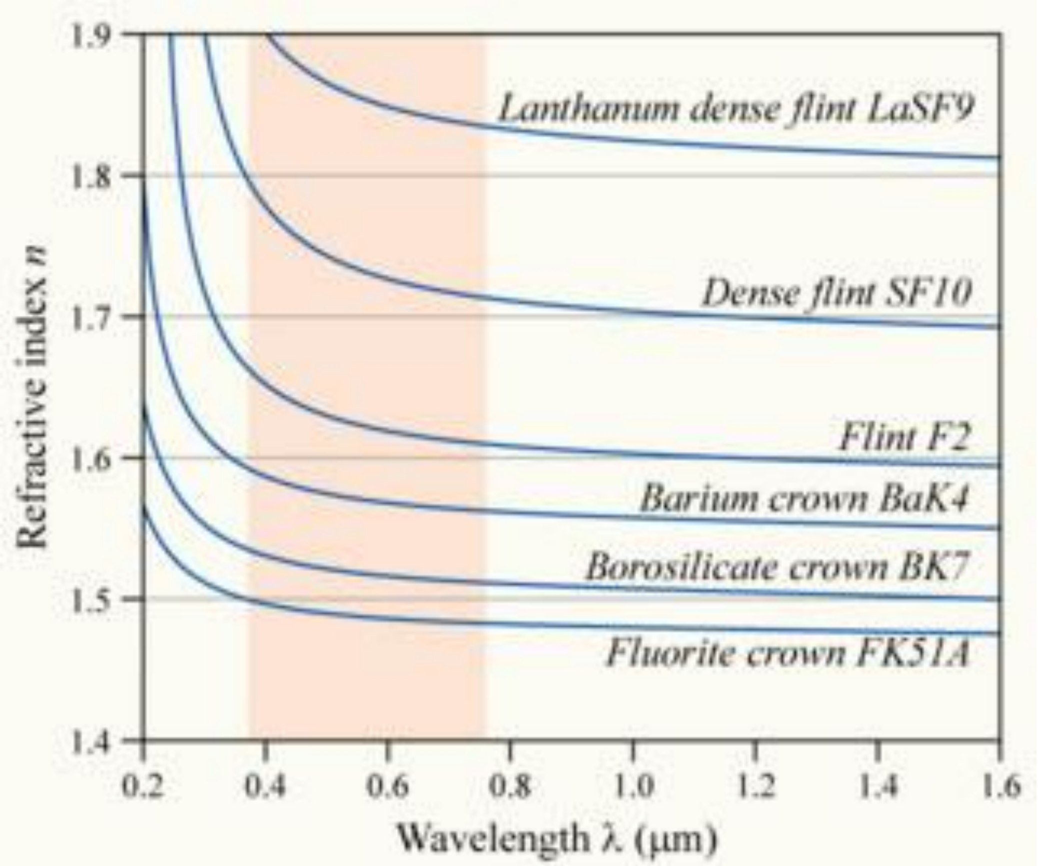
$$= v - \lambda \frac{dv}{d\lambda}$$

This term can be either positive or negative.

$\lambda \uparrow$ $v \uparrow$: $v_g < v$ (normal dispersion)

$\lambda \uparrow$ $v \downarrow$: $v_g > v$ (anomalous dispersion)

The dependence of refractive index to wavelength for various glasses



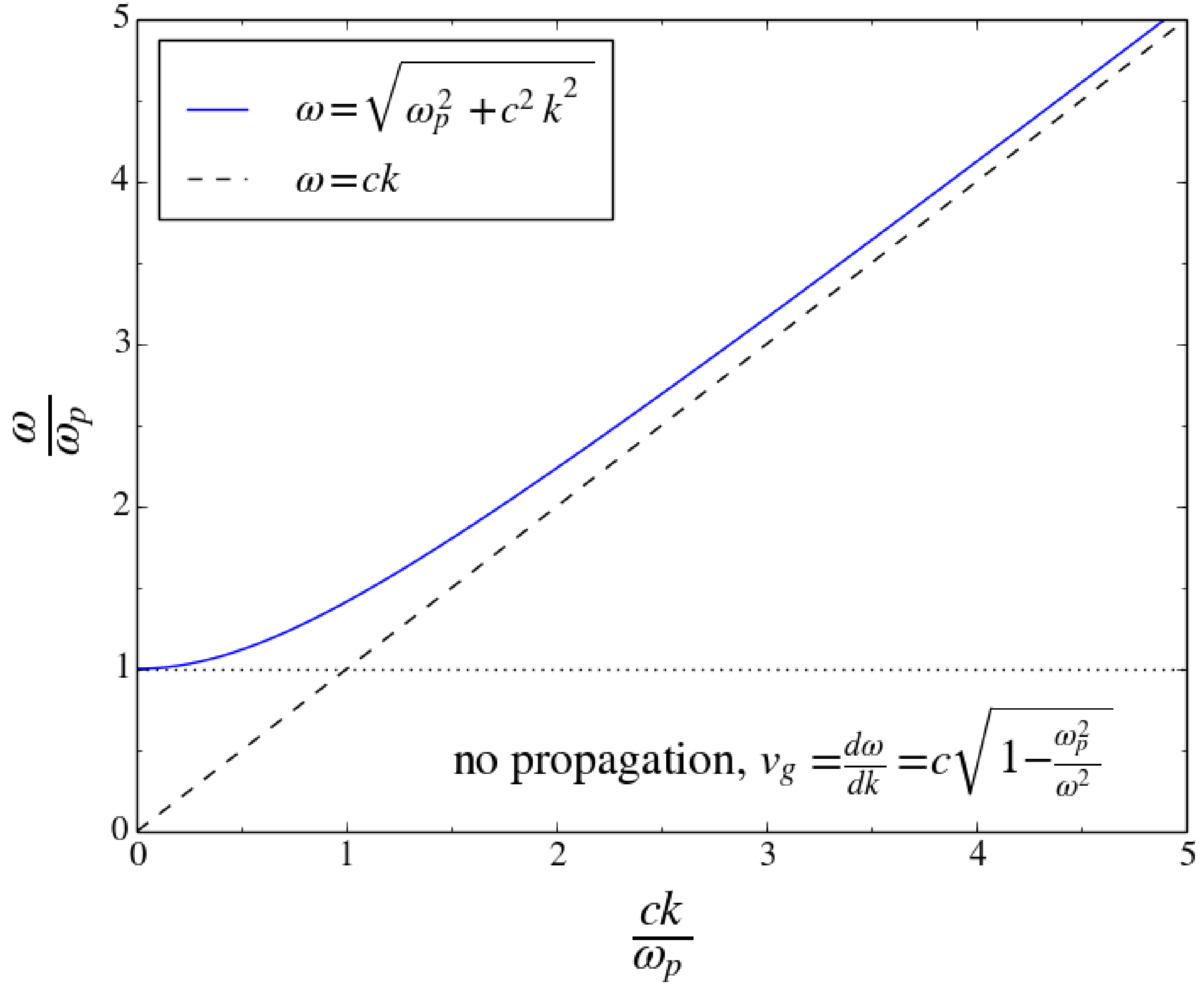
- **Are these various types of glasses non dispersive, normal dispersive or anomalous dispersive?**

Example 4

- It was found previously that the relative permittivity of an ionized gas is given by

$$\varepsilon_r = \frac{c^2}{v^2} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 ; \omega_p = plasma\ frequency$$

- This indicates that the relative permittivity is a function of frequency.
- Show that the dispersion relation is given as $\omega^2 = \omega_p^2 + c^2 k^2$



The dispersion relation for EM waves in a plasma

- What happens when $\frac{\omega}{\omega_p} < 1$?

Additional explanation

- Recall $\varepsilon_r = \frac{c^2}{v^2} = 1 - \left(\frac{\omega_p}{\omega}\right)^2$
- This actually relates to the refractive index of a medium as $n = \sqrt{\varepsilon_r}$.
- Once $\frac{\omega}{\omega_p} < 1$, the refractive index n becomes imaginary; i.e., $n = i\kappa$.
- Generally, the wave propagates in a medium is given as $E = E_0 e^{i(kx-\omega t)}$ and the wave number k can be written $k_0 n$.
- Because a general form of a refractive index is given by $n = \mathrm{N} + i\kappa$, the wave propagating inside the medium becomes

$$E = E_0 e^{i(kx-\omega t)} = E_0\, e^{-\kappa x}\, e^{i(k_0 N x-\omega t)}$$

This terms give rise to the attenuation of the amplitude as a function of the propagating distance.

Transverse waves in a periodic structures representing a 1D crystal structure with identical single atom separated by a crystal lattice *a*

- Consider the propagation of transverse waves along a linear array of atoms, mass m, in a crystal lattice where the tension **T represents the elastic force** between the atoms and *a* is the separation between atoms (so that T/*a* is the stiffness).
- The displacement of *r*th particle due to the transverse waves is given as

$$y_r = A_r e^{i(\omega t - kx)} = A_r e^{i(\omega t - kra)}$$

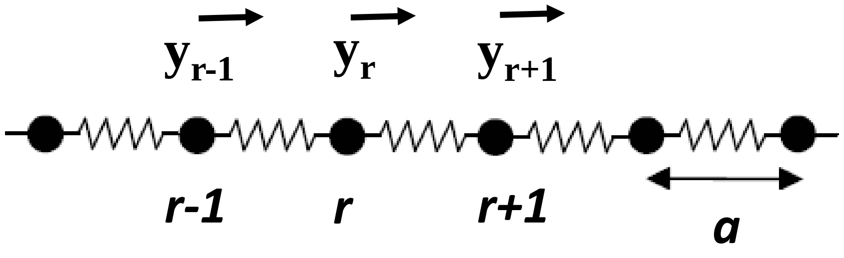
- By substituting the displacement into the equation of motion : $m\ddot{y}_r = \frac{T}{a}\left(y_{r+1} + y_{r-1} - 2y_r\right)$
- The **permitted frequencies** are found to be

$$\omega^2 = \frac{4T}{ma}\sin^2\frac{ka}{2}$$

This actually is the dispersion relation for waves travelling along a linear one dimensional array of atoms in a periodic structure.

An alternative way to derive the equation of motion in case of the one dimensional lattice vibration

- For simplicity, the forces between the atoms in a one-dimensional crystal lattice is assumed to be proportional to relative displacements from the equilibrium positions.



- Equation of motion for rth mass:

$$m\ddot{y}_r = -C\left(y_r - y_{r-1}\right) - C\left(y_r - y_{r+1}\right)$$

$$= C\left(y_{r+1} + y_{r-1} - 2y_r\right)$$

- Due to the stiffness C = T/*a*; therefore

$$m\ddot{y}_r = \frac{T}{a}\left(y_{r+1} + y_{r-1} - 2y_r\right)$$

What is the permitted frequency?

- The expression for ω^2 is equivalent to jth normal mode frequency of a transverse wave on loaded string,

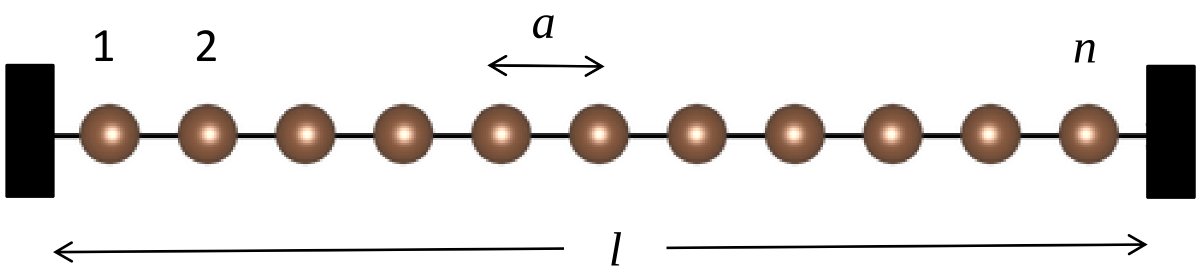
$$\omega_j^2 = \frac{2T}{ma}\left(1-\cos\frac{j\pi}{n+1}\right) = \frac{4T}{ma}\sin^2\frac{j\pi}{2(n+1)} \stackrel{?}{=} \omega^2 = \frac{4T}{ma}\sin^2\frac{ka}{2}$$

Permitted frequency

- If $\frac{ka}{2} = \frac{j\pi}{2(n+1)}$ (Show this?)

- Therefore, the permitted frequency is actually the normal mode frequency of the transverse wave existing in the linear array of atoms in a periodic structure.

Show $\frac{ka}{2} = \frac{j\pi}{2(n+1)}$



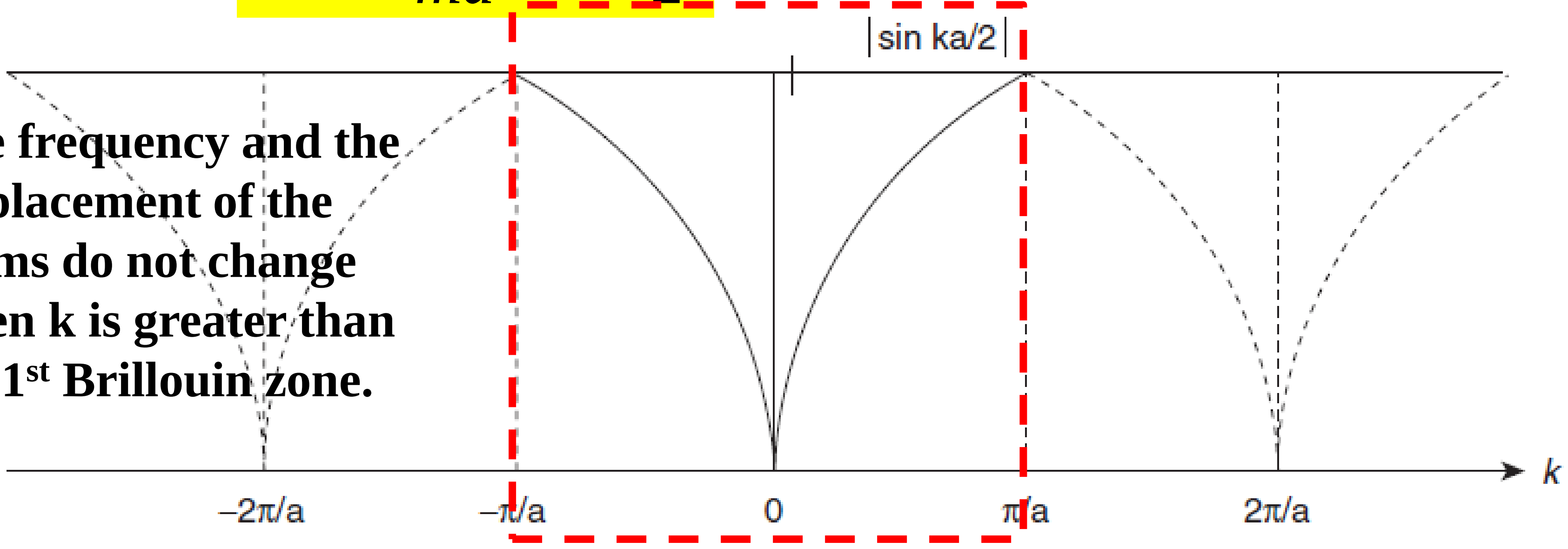
- Recall $(n+1)a$ = the length of the string or crystal, therefore $\frac{ka}{2} = \frac{j\pi a}{2(n+1)a} = \frac{j\pi a}{2l}$
- Also, the allowed wavelengths have to satisfy with $\frac{p\lambda}{2} = l$; where p is an integer.
- Thus, $\frac{ka}{2} = \frac{j\pi a}{2(n+1)a} = \frac{2j\pi a}{2p\lambda} = \frac{j\pi a}{p\lambda}$
- Because j and p are integers and they can be equal ($j = p$) to indicate a particular normal mode of interest. Provided that $k = 2\pi/\lambda$, we then can conclude that $\frac{ka}{2} = \frac{j\pi}{2(n+1)}$

- The **maximum permitted frequency** is found to be $\omega^2 = \frac{4T}{ma}$

- This gives $ka = \pi \rightarrow \lambda = 2a$ which is the **allowed minimum wavelength** for the transverse wave to propagate along the linear array of atoms in a periodic structure.
- The graph below shows the **dispersion relation (ω vs k)** for the linear array atoms.

- Due to $\omega^2 = \frac{4T}{ma}\sin^2\frac{ka}{2}$, the permitted frequency is then represented as $\left|\sin\frac{ka}{2}\right|$

The frequency and the displacement of the atoms do not change when k is greater than the 1st Brillouin zone.



1st Brillouin zone

Note that the repetition values of ω beyond the region $-\pi/a \leq k \leq \pi/a$, this region defines a **Brillouin zone**.

Dispersion relation for travelling long and short wavelength waves in a periodic structure

- For long wavelengths or low values of the wave number *k*, sin *ka*/2 → *ka*/2

$$\omega^2 = \frac{4T}{ma}\frac{k^2a^2}{4}$$

$$\therefore c^2 = \frac{\omega^2}{k^2} = \frac{Ta}{m} = \frac{T}{\rho}$$

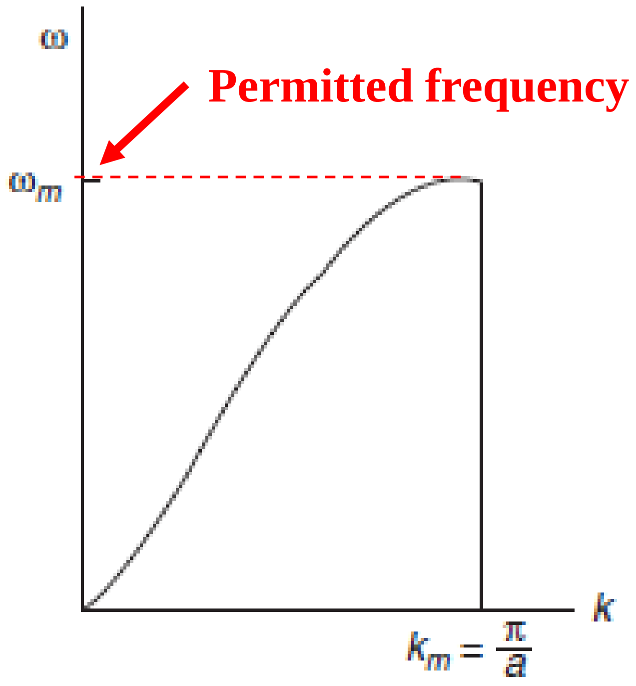
Wave velocity

- For short wavelengths, the phase velocity is given by

$$v = \frac{\omega}{k} = c\left[\frac{\sin ka/2}{ka/2}\right]$$

- Note : only at very short wavelengths does the atomic spacing of the crystal structure affect the speed of the wave propagation.

Dispersion relation for travelling waves in a periodic structure



- Recall the dispersion relation: $$\omega = \sqrt{\frac{4T}{ma}} \sin \frac{ka}{2}$$

- The maximum permitted frequency is obtained when

$$k = \pi/a \ \text{and} \ \lambda = 2a$$

- **Example** : the elastic force constant T/*a* for a crystal is about 15 Nm^{-1} and a typical reduced atomic mass is about 60 x 10^{-27} kg. **The maximum frequency is found to be about 5 x 10^{12} Hz**.

The maximum frequency is known as CUT OFF frequency.

Linear array of two kinds of atoms in an ionic crystal

- In this case, a one dimensional line which contains two kinds of atoms with separation *a* is considered.
- Given that atoms of mass ***M* occupying the odd numbered positions** and those of mass ***m* occupying the even numbered positioned**, the equation of motion for each type are

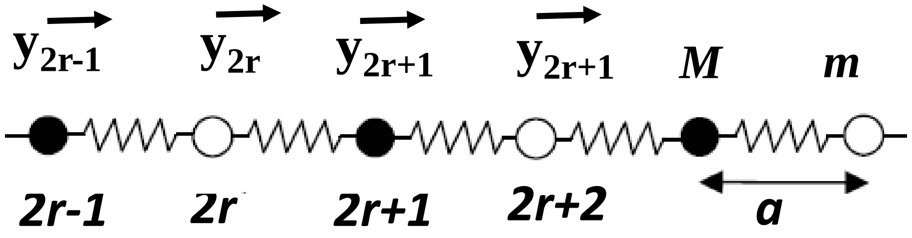
$$m\ddot{y}_{2r} = \frac{T}{a}\left(y_{2r+1} + y_{2r-1} - 2y_{2r}\right) \qquad M\ddot{y}_{2r+1} = \frac{T}{a}\left(y_{2r+2} + y_{2r} - 2y_{2r+1}\right)$$

- With solutions $y_{2r} = A_m e^{i(\omega t - 2rka)}$ and $y_{2r+1} = A_M e^{i(\omega t - (2r+1)ka)}$
- This gives the dispersion relation as

$$\omega^2 = \frac{T}{a}\left(\frac{1}{m} + \frac{1}{M}\right) \pm \frac{T}{a}\left[\left(\frac{1}{m} + \frac{1}{M}\right)^2 - \frac{4\sin^2 ka}{mM}\right]^{\frac{1}{2}}$$

An alternative way to derive the equation of motion in case of the diatomic one dimensional lattice vibration

- For simplicity, the forces between the atoms in a one-dimensional diatomic crystal lattice is assumed to be proportional to relative displacements from the equilibrium positions.



- Equation of motion for 2rth mass: $m\ddot{y}_{2r} = -C(y_{2r} - y_{2r-1}) - C(y_{2r} - y_{2r+1}) = C(y_{2r+1} + y_{2r-1} - 2y_{2r})$
- Equation of motion for (2r+1)th mass : $M\ddot{y}_{2r+1} = -C(y_{2r+1} - y_{2r}) - C(y_{2r+1} - y_{2r+2}) = C(y_{2r} + y_{2r+2} - 2y_{2r+1})$

- Due to the stiffness C = T/*a*; therefore $m\ddot{y}_{2r} = \frac{T}{a}(y_{2r+1} + y_{2r-1} - 2y_{2r})$

$$M\ddot{y}_{2r+1} = \frac{T}{a}(y_{2r} + y_{2r+2} - 2y_{2r+1})$$

https://unlcms.unl.edu/cas/physics/tsymbal/teaching/SSP-927/Section%2005_Lattice_Vibrations.pdf

Derivation of the dispersion relation

- From the equations of motion found earlier,

$$-\omega^2 m A_m = \frac{TA_M}{a}\left(e^{-ika} + e^{ika}\right) - \frac{2TA_m}{a}$$

$$-\omega^2 M A_M = \frac{TA_m}{a}\left(e^{-ika} + e^{ika}\right) - \frac{2TA_M}{a}$$

- By arranging the above equations in the following matrix form, the dispersion relation can be worked out.

$$\begin{bmatrix} \frac{2T}{a} - \omega^2 m & -\frac{2T}{a}\cos ka \\ -\frac{2T}{a}\cos ka & \frac{2T}{a} - \omega^2 M \end{bmatrix} \begin{bmatrix} A_m \\ A_M \end{bmatrix} = 0$$

Dispersion relation for the ionic crystal structure

$$\omega^2 = \frac{T}{a}\left(\frac{1}{m}+\frac{1}{M}\right) + \frac{T}{a}\left[\left(\frac{1}{m}+\frac{1}{M}\right)^2 - \frac{4\sin^2 ka}{mM}\right]^{\frac{1}{2}}$$

- Provided that m > M.
- Notice that the dispersion relation gives two possible situations.
- With the **positive sign**, the change of ω as a function of *k* is considered.
- The range of the wave number covers from $k = 0$ to $k_{max} = \pi/2a$ (minimum $\lambda = 4a$)

For $k = 0$ $\quad \rightarrow \omega^2 = \dfrac{2T}{a}\left(\dfrac{1}{m}+\dfrac{1}{M}\right)$

For k_{max} $\quad \rightarrow \omega^2 = \dfrac{2T}{aM}$

Note : ω_{max} is obtained when the minimum wavelength corresponds to $4a$ instead of $2a$ as found in the case of vibration in monatomic crystal structure).

Dispersion relation for the ionic crystal structure

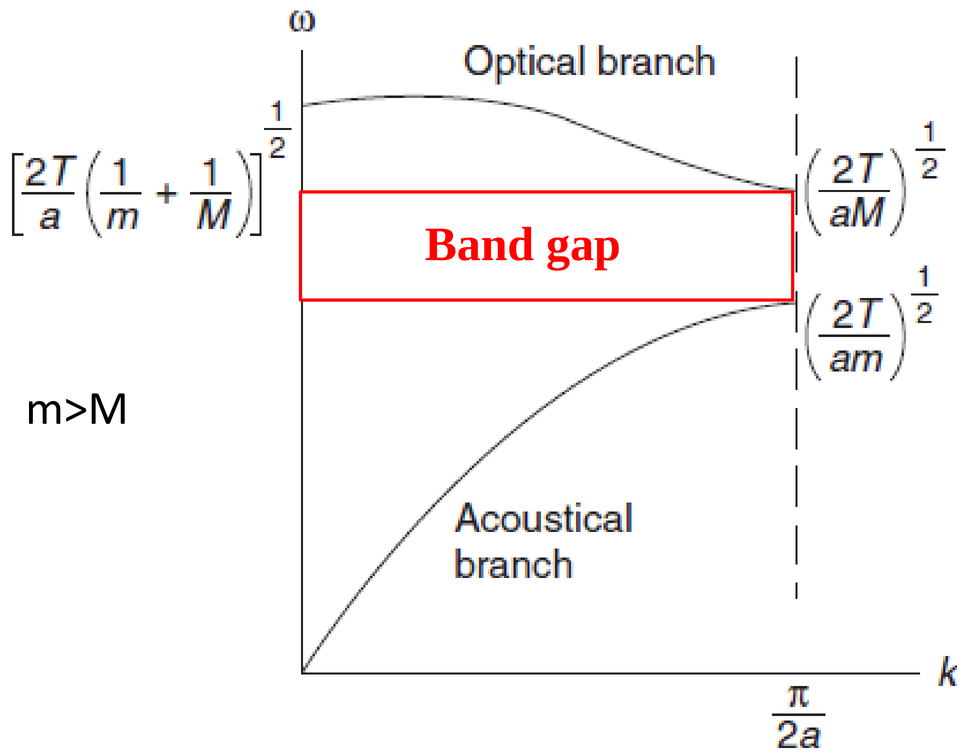
$$\omega^2 = \frac{T}{a}\left(\frac{1}{m}+\frac{1}{M}\right) - \frac{T}{a}\left[\left(\frac{1}{m}+\frac{1}{M}\right)^2 - \frac{4\sin^2 ka}{mM}\right]^{\frac{1}{2}}$$

- Provided that m > M.
- Notice that the dispersion relation gives two possible situations.
- With the **negative sign**, the change of ω as a function of *k* is considered.
- The range of the wave number covers from $k = 0$ to $k_{max} = \pi/2a$ (minimum $\lambda = 4a$)

For $k = 0$ $\rightarrow \omega = 0$; For small k $\rightarrow \omega^2 = \dfrac{2Tk^2a^2}{a(M+m)}$

For k_{max} $\rightarrow \omega^2 = \dfrac{2T}{am}$

Dispersion relation for two modes of transverse oscillation in a crystal structure

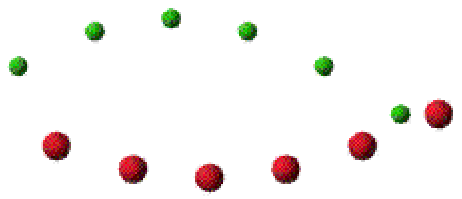


m>M

$$\omega^2 = \frac{T}{a}\left(\frac{1}{m}+\frac{1}{M}\right) \pm \frac{T}{a}\left[\left(\frac{1}{m}+\frac{1}{M}\right)^2 - \frac{4\sin^2 ka}{mM}\right]^{\frac{1}{2}}$$

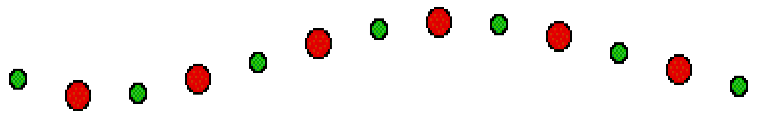
- According to the dispersion relation, two possible solutions can be found for m > M.
- They are **optical mode (upper branch)** and **acoustic mode (lower branch)**.
- Also there is a forbidden band (band gap) of frequencies between the two branches that the wave cannot propagate.
- **The band gap depends on the differences of masses.**

The motions of the two types of atom for each branch at k → 0 (long wavelength)



Optical mode for long wavelength and small *k*, **A_m/A_M = -M/m**

The atoms vibrate against each other, the center of mass of the unit cell in the crystal remains fixed.



Acoustic mode for long wavelength and small *k*, **$A_m = A_M$**

Atoms and their center of mass move together.

http://slideplayer.com/slide/9413111/

Derivation of the motions of the two types of atom for each branch

- Recall the equation of motion for the ionic crystal structure previously discussed,

$$\begin{bmatrix} \frac{2T}{a}-\omega^2 m & -\frac{2T}{a}\cos ka \\ -\frac{2T}{a}\cos ka & \frac{2T}{a}-\omega^2 M \end{bmatrix}\begin{bmatrix} A_m \\ A_M \end{bmatrix}=0$$

- The relative motion between *m* and *M* with a small *k* can be found when suitable frequency for each branch is considered.

Optical branch

- Using one of the two equations from the matrix

$$\left(\frac{2T}{a} - m\omega^2\right)A_m - A_M \frac{2T}{a}\cos ka = 0$$

- For k = 0 $\rightarrow$ $\omega^2 = \frac{2T}{a}\left(\frac{1}{m} + \frac{1}{M}\right)$

- The relative motion is found to be

$$\therefore \frac{A_M}{A_m} = -\frac{m}{M}$$

Acoustic branch

- Using one of the two equations from the matrix

$$\left(\frac{2T}{a} - m\omega^2\right)A_m - A_M \frac{2T}{a}\cos ka = 0$$

- For k = 0 $\rightarrow$ $\omega^2 = 0$

- The relative motion is found to be

$$\therefore \frac{A_M}{A_m} = 1$$

Example 5
Absorption of infrared radiation by ionic crystal

- Suppose the ionic crystals composed of ions of opposite charges $\pm e$ move under the influence of the electric field $E = E_0 \exp(i\omega t)$.
- The equations of motion can be written as

$$m\ddot{y}_{2r} = \frac{T}{a}\left(y_{2r+1} + y_{2r-1} - 2y_{2r}\right) - eE; \text{ for negative ion}$$

$$M\ddot{y}_{2r+1} = \frac{T}{a}\left(y_{2r} + y_{2r+2} - 2y_{2r+1}\right) + eE; \text{ for positive ion}$$

- Appropriate displacement for each ion is given as follows

$$y_{2r} = A_m e^{i\left(\omega t - 2rka\right)} \quad \text{and} \quad y_{2r+1} = A_M e^{i\left(\omega t - \left(2r+1\right)ka\right)}$$

- However, the analysis can be simplified by neglecting the wave number *k* if we consider the situation at a longer wavelength.
- Assuming that the electric field is infrared and its wavelength is around 10^{-4} m.
- This gives the wave number $k \approx 6\times10^{4}$ m^{-1}. This value is negligible when comparing to the k_m which is the limit of wave number at the boundary of 1st Brillouin zone. Approximately, k_m is found to be $\pi/2a$ where $a = 10^{-10}$ m.
- Under the assumption, this can simplify the equations of motion for each ion to be

$$-\omega^2 m A_m = \frac{2T}{a}\left(A_M - A_m\right) - eE_0$$

$$-\omega^2 M A_M = \frac{-2T}{a}\left(A_M - A_m\right) + eE_0$$

- The amplitudes for both ions are found to be

$$A_m = \frac{-eE_0}{m\left(\omega_0^2 - \omega^2\right)}$$

$$A_M = \frac{eE_0}{M\left(\omega_0^2 - \omega^2\right)}$$

- Where $$\omega_0^2 = \frac{2T}{a}\left(\frac{1}{m} + \frac{1}{M}\right)$$

This actually is the low k limit of the optical branch.

- When $\omega = \omega_0$, the ions amplitude increases meaning a strong absorption by ionic crystals

Homework #6

Problem 5.16

– The dielectric constant of a gas at a wavelength λ is given by

$$\varepsilon_r = \frac{c^2}{v^2} = A + \frac{B}{\lambda^2} - D\lambda^2$$

where A, B and D are constants, c is the velocity of light in free space and v is its phase velocity. If the group velocity is V_g show that

$$V_g \varepsilon_r = v(A - 2D\lambda^2)$$

Problem 5.25

An aircraft flying on a level course transmits a signal of 3×10^9 Hz which is reflected from a distant point ahead on the flight path and received by the aircraft with a frequency difference of 15 kHz. What is the aircraft speed?

Problem 5.26

Light from hot sodium atoms is centred about a wavelength of 6×10^{-7} m but spreads 2×10^{-12} m on either side of this wavelength due to the Doppler effect as radiating atoms move towards and away from the observer. Calculate the thermal velocity of the atoms to show that the gas temperature is ~ 900 K.